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EIGENVALUES, EIGENVECTORS AND STATE-SPACE MATHEMATICS

Theory, Mathematical Foundations, Step-by-Step Examples, Visual Interpretation, and EE/ECE Applications

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ABSTRACT

Eigenvalues reveal natural modes, stability and resonance. Eigenvectors identify directions that a linear transformation scales without rotating. This paper develops the subject from first principles, explains how to read every principal equation, demonstrates numerical use, and connects the mathematics to practical electrical and electronics systems.

LEARNING OUTCOMES

After studying this paper, a learner should be able to: (1) explain eigenvalue equation, diagonalisation, state-space model, stability; (2) identify variables, units and assumptions; (3) sketch the expected trend before calculation; (4) solve representative problems step by step; and (5) select the correct method for an EE/ECE application.

SCOPE AND PREREQUISITES

The discussion begins at a beginner level and progresses to the foundation expected in B.Tech and introductory M.Tech electrical, electronics, communication, control, embedded and semiconductor courses. School-level arithmetic and basic equation manipulation are sufficient prerequisites.

HOW TO USE THIS PAPER

Read the physical explanation first. Identify every symbol and unit. Reproduce the worked examples without viewing the solution. Change one value, predict the direction of the result, and then recalculate.

Document code: CAEE-MATH-06 | Detailed learning edition | Generalized CAEE educational material.

I. WHY THIS MATHEMATICS MATTERS

Eigenvalues reveal natural modes, stability and resonance. Eigenvectors identify directions that a linear transformation scales without rotating.

Mathematics is a model of cause and effect. Independent variables describe what is changed or observed; dependent variables describe the response; parameters describe the device or system. A valid result must satisfy the equation, units, expected sign and physical limits.

II. NOTATION AND READING RULES

Italic symbols usually represent variables. Bold symbols represent vectors or matrices. Subscripts identify components, samples or conditions. Superscripts may indicate a power, transpose or inverse. Parentheses identify a function argument.

Beginner rule: establish the symbolic relationship and required units before substituting numbers.

III.1 EIGENVALUE EQUATION

An eigenvector keeps its direction after transformation. The characteristic equation finds the permitted scaling factors.

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{1}$$

How to read it. Identify the output first, then the inputs and operation. State which quantities vary and which are fixed parameters.

Checks. Test zero, unity and a known boundary. Predict the effect of doubling the dominant parameter.

SYMBOL-BY-SYMBOL ANALYSIS

1. Read the left-hand side as the quantity being defined or calculated.
2. Separate constants, independent variables and dependent variables on the right-hand side.
3. Write the unit beside every physical quantity.
4. Identify operations that change scale, direction, rate, phase or probability.
5. State the range over which the expression is valid.

LOCAL SENSITIVITY

For eigenvalue equation, vary one input slightly while the remaining parameters are fixed. If the output changes by the same fraction, the local relationship is proportional. A much larger output change indicates high sensitivity; a smaller change indicates compression or saturation.

PHYSICAL REASONABLENESS

Estimate the answer before calculating. A result that is many orders of magnitude away from the expected circuit, signal, field or statistical scale normally indicates a unit-prefix, sign or model-selection error.

III.2 DIAGONALISATION

When enough independent eigenvectors exist, a matrix can be expressed in modal coordinates. Powers and exponentials then become easier to compute.

$$\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1} \tag{2}$$

How to read it. Follow the equation from left to right and identify each mathematical operation. Confirm whether sign, order or direction changes the result.

Checks. Substitute a simple reference case and compare the result with the graph or physical expectation.

FORMULA-TO-GRAPH CONNECTION

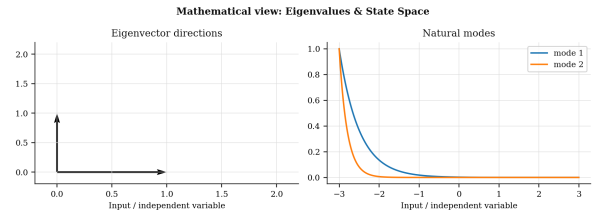


Fig. 1. Mathematical behaviour associated with Eigenvalues & State Space. The axes use a fixed scale for meaningful comparison.

A formula states the relationship exactly; a graph reveals its behaviour. Inspect intercepts, slope, curvature, symmetry, discontinuities, asymptotes and steady-state values. These features help detect sign or scaling errors before a detailed calculation.

GRAPH-READING EXERCISE

Trace the curve from left to right. Mark a reference point, a rapidly changing region and a limiting region. Explain which equation term dominates in each region. Then imagine increasing the principal parameter and predict whether the curve moves vertically, horizontally, rotates, widens or changes curvature.

EQUATION AND GRAPH CONSISTENCY

The sign of the derivative should agree with whether the graph rises or falls. Symmetry in the equation should appear as symmetry in the plot. A pole, zero, discontinuity or boundary should have a visible consequence. If the graph and equation disagree, verify scaling and implementation before trusting either one.

TOOL CONNECTION

A calculator returns a number, while MATLAB, Python, a spreadsheet or a circuit simulator can sweep a parameter. Use both: verify one point by hand and use the sweep to understand the full behaviour. Keep the axes fixed so visual rescaling does not create a false impression of change.

PAGE 2 REVIEW

At this stage, the learner should be able to define eigenvalue equation and diagonalisation, reproduce equations (1) and (2), and explain the visible trends without relying on memorised words.

IV.1 STATE-SPACE MODEL

State variables capture stored energy and memory. Inputs drive the states and outputs are measured combinations of states and inputs.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3}$$

Interpretation. The formula is a compact statement of behaviour, not only a calculation instruction. Its graph may reveal monotonicity, curvature, oscillation, discontinuity, steady-state value or sensitivity.

Assumptions. Check the domain, sign convention, continuity requirements, initial or boundary conditions, and whether the model is ideal or approximate.

DERIVATION PATHWAY

Begin with the defining relationship, apply the relevant algebraic or calculus operation, then use symmetry or physical conditions to remove impossible terms. A derivation is complete only when every introduced symbol is defined and every step preserves units.

DERIVATION AUDIT

• Each equality must follow from the preceding line.
• A cancellation is permitted only when the cancelled quantity is nonzero.
• Squaring, taking roots or applying inverse functions can introduce additional branches or restrictions.
• Initial and boundary conditions select constants only after the general form is established.
• An approximation must state the small or large parameter that justifies it.

DIMENSIONAL CHECK

Both sides of an equation must have compatible dimensions. Dimensionless arguments are required inside exponential, logarithmic and trigonometric functions.

ENGINEERING UNIT EXAMPLE

Convert prefixes before substitution: milli = 10^-3, micro = 10^-6, nano = 10^-9, kilo = 10^3 and mega = 10^6. Distinguish hertz from radians per second using omega = 2 pi f. State whether amplitudes are peak, peak-to-peak or RMS.

NUMERICAL PRECISION

Retain extra digits during intermediate calculations and round only the reported result. Precision does not correct an inaccurate model, but premature rounding can hide cancellation or shift a stability boundary.

IV.2 STABILITY

A continuous-time linear system is asymptotically stable when all eigenvalues of A have negative real part.

$$\Re\{\lambda_i\} < 0 \tag{4}$$

Interpretation. Relate each symbol to a measurable quantity or a clearly defined mathematical object. State whether the result is scalar, vector, matrix, function, probability or transform.

Assumptions. Verify stability, convergence, independence, linearity or smoothness when the selected method depends on those conditions.

V. CONNECTIONS BETWEEN THE CONCEPTS

Eigenvalue equation establishes the primary representation. Diagonalisation supplies an important operation or property. State-space model extends the model, while Stability connects it to engineering analysis. A strong solution states this chain explicitly.

LIMITING-CASE TEST

Evaluate the input approaching zero, a parameter becoming very large, time approaching infinity, or frequency approaching a boundary. An answer that violates a known limit is wrong even when produced by a calculator.

MODEL VALIDITY MAP

An ideal model usually neglects loading, parasitics, saturation, quantisation, finite bandwidth, noise or manufacturing spread. List what has been neglected and predict the direction of the resulting error. For example, loss normally reduces an ideal magnitude, finite bandwidth rounds fast transitions, and noise introduces uncertainty around a deterministic prediction.

CONCEPT CHECK

1. Which assumption is most important for stability?
2. Which parameter determines the dominant scale?
3. Which equation term controls the limiting behaviour?
4. What measurement would distinguish an ideal model from a real implementation?

EXPERT HABIT

Maintain a short decision log: model selected, alternative rejected, assumption used, unit convention and verification method. This makes the analysis reproducible and exposes hidden reasoning errors.

PAGE 3 REVIEW

Explain how state-space model and stability extend the first two ideas. State at least one assumption and one limiting case for each displayed formula.

VI. MATHEMATICAL AND PHYSICAL VISUALISATION

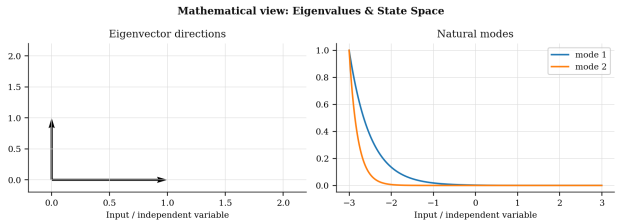


Fig. 2. Complementary mathematical views of Eigenvalues & State Space.

First locate intercepts, extrema and asymptotes. Next decide whether the response increases, decreases, oscillates or approaches a steady value. Finally connect each feature to a physical cause such as stored energy, phase displacement, spatial direction, randomness, feedback or numerical approximation.

VII. ENGINEERING PROBLEM-SOLVING METHOD

1 - Define the system. Draw the circuit, signal, field, algorithm or data set.
2 - List variables and units. Use one unit system and declare the sign convention.
3 - Select the model. Write the governing equation before inserting numbers.
4 - Simplify symbolically. Apply symmetry, initial conditions or identities.
5 - Calculate. Show intermediate values with sufficient precision.
6 - Verify. Check units, limits, magnitude, graph shape and plausibility.

DESIGN QUESTION

Which parameter is most influential? Change one quantity at a time, plot the response on fixed axes and compare the percentage change in output.

VIII. EE/ECE APPLICATIONS

1. Control Stability. Convert measurable voltage, current, frequency, time, geometry or probability into a quantity that can be designed or verified.

2. Rlc Natural Modes. Predict how a parameter change moves gain, phase, energy, stability, error or bandwidth.

3. Power-System Oscillations. Use the equations behind simulation tools to select settings and recognise an incorrect result.

4. Pca And Modal Analysis. Compare laboratory measurements with the ideal model while accounting for tolerance, noise, loading and sampling.

IX. MODEL, SIMULATION, AND MEASUREMENT

Model. List ideal assumptions and calculate a hand estimate.
Simulation. Sweep the dominant parameter and verify limiting cases.
Measurement. Record instrument settings, units, loading and uncertainty.
Reconciliation. Explain differences using omitted physics, parasitics, tolerance or measurement limitations.

SPECIAL-CASE ANALYSIS

For eigenvalues & state space, test a zero input, a unity parameter, equal variables, very large or small time/frequency, and the boundary between valid and invalid operation. A familiar special case is a powerful algebra check.

APPLICATION SCENARIO

Consider a design involving control stability and RLC natural modes. Start with the ideal expression, identify the parameter under designer control, and calculate a nominal value. Next include one non-ideal effect such as tolerance, finite bandwidth, noise, loading or numerical resolution. The difference between these two results explains why mathematical understanding is required even when simulation software is available.

SIMULATION CHECKLIST

- Fix graph axes before comparing parameter sweeps.
- Label every trace, axis and unit.
- Save the nominal, boundary and limiting cases.
- Compare one plotted point with a hand calculation.
- Separate numerical artefacts from physical behaviour.

MODEL-TO-MEASUREMENT ERROR

$$\varepsilon_{\%} = \frac{|x_{\text{measured}} - x_{\text{model}}|}{|x_{\text{model}}|} \times 100\% \quad (8)$$

A percentage difference is useful only when the reference is nonzero and the same quantity, bandwidth, normalization and operating condition are used. A mismatch should be investigated, not automatically labelled as measurement error.

X. WORKED EXAMPLES

Use the sequence: given information, governing relation, unit conversion, substitution, calculation and physical interpretation.

EXAMPLE 1 - FOUNDATION

For A=diag(-2,-5), the modes decay as e^(-2t) and e^(-5t); the slower -2 mode dominates settling.

Governing textbook relation.

$$\Re\{\lambda_i\} < 0 \tag{5}$$

Check the final unit, sign and order of magnitude. Change the principal parameter and verify that the result moves in the direction predicted by the relation.

EXAMPLE 2 - RLC POLES

Given. A second-order system has poles -2 plus or minus j3.

$$x_n(t) = e^{-2t}(C_1 \cos 3t + C_2 \sin 3t) \tag{6}$$

Interpretation. The real part controls decay and the imaginary part controls oscillation.

NUMERICAL DISCIPLINE

Carry symbols until the governing relationship is clear. Convert engineering prefixes explicitly and round only the final result.

EXAMPLE 3 - DISCRETE STABILITY

Given. State matrix eigenvalues are 0.7 and 1.05.

$$\rho(\mathbf{A}) = \max_i |\lambda_i| = 1.05 > 1 \tag{7}$$

Interpretation. A discrete-time state model is unstable because one modal magnitude exceeds unity.

XI. SENSITIVITY AND TOLERANCE

Vary one parameter by +10 percent while holding the others constant. Compare the percentage change in the result. A proportional law gives approximately the same percentage change; a squared law gives approximately twice the change for small variations; an exponential law may be much more sensitive.

$$S_y^p = \frac{p}{y}\frac{\partial y}{\partial p} \tag{9}$$

The dimensionless sensitivity S describes the fractional change in output y produced by a fractional change in parameter p. A magnitude greater than one identifies an amplified tolerance; a magnitude below one indicates reduced sensitivity. The sign states whether the output rises or falls.

For several component tolerances, calculate nominal, best-case and worst-case values before using statistical methods. In feedback, resonance, probability and numerical problems, a small change can cross a qualitative boundary, so a sweep is safer than one evaluation.

UNCERTAINTY PROPAGATION

$$u_y \approx \sqrt{\sum_i \left(\frac{\partial y}{\partial x_i} u_{x_i}\right)^2} \tag{10}$$

This root-sum-square approximation is appropriate for small, independent input uncertainties and a locally smooth model. Correlated variables require covariance terms; strongly nonlinear behaviour requires direct sampling or Monte Carlo analysis.

XII. GUIDED PRACTICE

1. Rework Example 1 with its principal parameter increased by 25 percent.
2. Construct a realistic example using diagonalisation.
3. Sketch the result for state-space model before calculating.
4. Explain the effect of measurement tolerance or rounding.
5. State one condition under which the model is invalid.

SOLUTION CHECKLIST

Include a system description, given quantities with units, symbolic equation, substitution, intermediate calculation, final value with unit, and one sentence of physical interpretation.

XIII. FORMULA SUMMARY

Concept	Textbook form
Eigenvalue equation	$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}, \qquad \det(\mathbf{A} - \lambda\mathbf{I}) = 0$
Diagonalisation	$\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$
State-space model	$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$
Stability	$\Re\{\lambda_i\} < 0$

XIV. COMMON MISTAKES AND CORRECTIONS

• Formula matching: define the physical model first.
• Premature substitution: simplify symbolically before inserting values.
• Unit inconsistency: convert prefixes and angular/ordinary frequency explicitly.
• Missing conditions: retain initial values, boundaries, convergence, stability or independence assumptions.
• Calculator trust: verify sign, magnitude and limiting behaviour.
• Graph misreading: label axes and distinguish amplitude, power, probability and logarithmic scales.

XV. VERIFICATION TABLE

Check	Question
Units	Do both sides have compatible dimensions?
Sign	Does direction or polarity agree with the convention?
Limit	Does zero or infinity give the expected behaviour?
Magnitude	Is the order of magnitude physically realistic?
Graph	Does the calculated point lie on the expected trend?

XVI. B.TECH AND M.TECH EXPECTATIONS

Level	Expected capability
B.Tech	Define symbols, solve standard problems, sketch graphs and connect equations to circuits or signals.
Advanced B.Tech	Derive principal relations, analyse non-ideal cases and verify simulations using hand calculations.
M.Tech	Evaluate assumptions, sensitivity, stability, convergence, uncertainty and numerical implementation.

XVII. SELF-ASSESSMENT

• Explain eigenvalue equation in your own words.
• Identify the assumptions behind all four equations.
• Rework every numerical example with one parameter doubled.
• Name two applications from your EE/ECE specialisation.
• Sketch the expected graph before calculating.
• Justify one formula from its definition.
• Describe a practical measurement that verifies the model.

XVIII. KEY TAKEAWAYS

Eigenvalues & State Space should be understood at four levels: symbols, mathematical operation, graphical behaviour and physical interpretation. Preserve units, state assumptions, solve step by step and verify the answer against engineering reality.

XIX. ORAL-EXAM PROMPTS

1. What physical meaning does the principal equation carry?
2. Which assumption is most likely to fail in a real control stability application?
3. Which parameter dominates sensitivity?
4. How would you validate the result experimentally?
5. What changes when the problem moves from B.Tech calculation to M.Tech analysis?

XX. GLOSSARY

Eigenvalue equation: primary representation.
Diagonalisation: supporting operation or property.
State-space model: extended behaviour.
Stability: engineering interpretation.
Parameter: value characterising a model.
Variable: quantity allowed to change.
Condition: information selecting the physical solution.
Sensitivity: output change caused by an input change.

XXI. STUDY ROADMAP

1. Copy and label the four main equations.
2. Reproduce the three examples without viewing the answer.
3. Draw the principal graph on fixed axes.
4. Implement one calculation in software.
5. Compare theory with one EE/ECE application.
6. Explain the topic verbally in two minutes.

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